Dynamics for Mechanical Engineering Majors

David Ng

Fall 2017

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§1 September 11, 2017

§1.1 Introduction

Class wasted.

§2 September 13, 2017

§2.1 Rectilinear Motion of a Particle

We begin by introducing (again) definitions. A *position* is a vector in general. Given a direction $\vec{r}$ and a time t, then the position P at time t is $P = \vec{r}t$. The *velocity* is also a vector. Denoted $\vec{v}$, it is given by $\vec{v} = \lim_{\Delta t \to 0} \Delta\vec{r}/\Delta t = \mathrm{d}r/\mathrm{d}t$. The *speed* is given as the magnitude of the velocity, and is $\|\vec{v}\|$. The *acceleration* $\vec{a}$ is the derivative of velocity, and is given by $\vec{a} = \mathrm{d}\vec{v}/\mathrm{d}t = \mathrm{d}^2\vec{r}/\mathrm{d}t^2$.

In the case of rectilinear motion, we have only one dimension to consider. The above expressions for velocity and acceleration can therefore be generalized in the following manner. In the special case for rectilinear motion, we have

$$v = \frac{\mathrm{d}x}{\mathrm{d}t},$$

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}x}.$$

For the special cases where $v = 0$ and $a = 0$, the equations can be simplified further. *Uniform rectilinear motion* means that v is constant, and so $a = 0$. To find the position in this case, we note that

$$x - x_0 = \int_{t_0}^{t} v\mathrm{d}t = v(t - t_0).$$

Uniformly accelerated rectilinear motion on the other hand, means that a is a constant. Thus, at an initial t_0 and v_0, we can integrate the expressions of velocity and acceleration to obtain

$$v - v_0 = \int_{t_0}^{t} a\mathrm{d}t = a(t - t_0),$$

$$x - x_0 = \int_{t_0}^{t} v\mathrm{d}t = v_0(t - t_0) + \frac{a(t - t_0)^2}{2},$$

$$v^2 = v_0^2 + 2a(x - x_0).$$

Example 2.1

A motorist enters a freeway at 25 miles/hour and accelerates uniformly to 65 miles/hour. From the odometer in the car, she knows that she has travelled 0.1 miles while accelerating. Determine the acceleration of the car, and the time required to reach 65 miles/hour.

Solution. We solve for a in the equation $v^2 = v_0^2 + 2a(x - x_0)$. Using $v = 65$, $v_0 = 25$, $x - x_0 = 0.1$, we find that the acceleration is therefore 18000 miles/hour. Now, using $v = v_0 + a(t - t_0)$, we can substitute known values to solve for $\Delta t = t - t_0$. Doing so, we find that $\Delta t = 0.0022$ hours. Therefore, the time required is approximately 8 seconds. ■

§3 September 15, 2017

§3.1 Relative Motion of Two Particles

Consider the rectilinear motion of two particles A and B. Oftentimes, we are concerned with the position of particle B with respect to (relative to) the position of particle A. We define the relative motion of particle B relative to particle A in terms of position, velocity, and acceleration:

$$x_{B/A} = x_B - x_A,$$

$$v_{B/A} = v_B - v_A,$$

$$a_{B/A} = a_B - a_A.$$

Remark 3.1. Note that when we do not consider relative motion, we still consider the motion of a particle relative to the arbitrary origin that we have set. Thus, relative motion can be thought of as fixing the origin at the location of a particle A, and considering the motion of B *relative* to A.

§3.2 Dependent Motion and Constraint Equations

There are times when the position of a particle depends on those of another particle. *Dependent motion* describes this phenomenon by attaching certain *constraints* on the position of two particles. When the relation between the position coordinates of particles in linear, then similar relations hold for velocity and acceleration.

Example 3.2 (Dependent Motion)

Suppose we have a pulley of radius R with masses A and B fixed on opposing sides of an inextensible cable. The center of the pulley is h from the ceiling, while A and B are x_A and x_B from the ceiling respectively. (This is not really a question).

Solution. Since we have two particles, we have two degrees of freedom. x_A and x_B are dependent, so we are left with one degree of freedom. Now, we need to find a constraint equation. Since we assume that the cable is inextensible, the length of the cable does not change, and thus we have fixed the remaining degree of freedom.

The left length of cable is $x_A - h$, the length of the cable curled around the top half of the pulley is πR (for the arc length), and the right length of cable is $x_B - h$. Our constraint equation is therefore $(x_A - h) + \pi R + (x_B - h) = C_1$, where C_1 is a constant. Since h and R are fixed, we can rewrite this as $x_A + x_B = C_2$, where C_2 is another constant. The displacement is therefore $\Delta x_A + \Delta x_B = 0$. We can find the derivative of both sides to obtain $v_A + v_B = 0$ and $a_A + a_B = 0$. ■

Remark 3.3. Note that there was no need to include h and πR in the equation, as it has no influence on Δx, v, and a. A fixed length difference can be ignored in the constraint equation as far as the motion is concerned. The relative velocity in this situation is $v_{B/A} = v_B - v_A = -2v_A$ since $v_A + v_B = 0$ means that $v_B = -v_A$.

Example 3.4

Now, suppose we have three particles. From the cable end fixed to the ceiling, we encounter the first pulley with mass A attached, the second pulley, the third pulley with mass B attached, the fourth pulley, and then finally mass C at the end of the cable. The second and fourth pulleys are attached to the ceiling at a fixed height h, while the first and third pulleys with A and B are free to move up and down. Determine the constraint equation of the pulley system.

Solution. We obtain constraint equations from the fact that the length of cable is fixed. Using reasoning similar to the first example above, we ignore the height of the pulleys and the arc length (assuming they are constant) to obtain the constraint $2x_A + 2x_B + x_C = C$, where C is a constant. Deriving the expression, we obtain $2v_A + 2v_B + v_C = 0$ and $2a_A + 2a_B + a_C = 0$. ■

Example 3.5

Block A moves down with a constant velocity of 1 m/s. Determine the velocity of block C, the velocity of collar B relative to block A, and the relative velocity of portion D of the cable with respect to block A.

Solution. We have three particles and two constraints. Since we are given the velocity of block A, we have zero degrees of freedom. First, the length of cable supporting A is constant, so $x_A + (x_A - x_B) = C_1$. Thus, taking the derivative, we obtain $2v_a - v_B = 0$. However, since we know that $v_A = 1$ m/s downwards, we know that $v_B = 2$ m/s downwards. For the cable supporting B, we have $2x_B + x_C = C_2$. The derivative gives $2v_B + c_V = 0$, so we find that $v_C = 4$ m/s upwards since we know what v_B is.

The relative velocity is $v_{B/A} = v_B - v_A = 2 - 1 = 1$ m/s. Since $v_B > v_A$, this means that B will catch up to A.

To find $v_{D/A}$, we need to find v_D. Point D is fixed on the cable, so we say that $x_D + x_C = 0$. Taking the derivative to be equal to 0, we find that $v_D = 4$ m/s downwards. Thus, $v_{D/A} = v_D - v_A = 4 - 1 = 3$ m/s. ■

§4 September 18, 2017

§4.1 Curvilinear Motion

Recall the *triangle law of vectors.* The resultant vector is simply the sum of all vectors. We denote $\vec{r}(t)$ indicates the position of a particle at time t. Then, $\Delta\vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t)$ from the triangle law. However, $\mathrm{d}\vec{r} = \lim_{\Delta t \to 0} \Delta\vec{r}$, so $\Delta\vec{r}$ and $\mathrm{d}\vec{r}/\mathrm{d}t$ are not necessarily in the same direction as $\vec{r}$.

For a particle moving in curved motion from point P to P', over a change in time from t to $t + \Delta t$, its change in position $\vec{r}$ is given as $\Delta\vec{r} = \vec{r}' - \vec{r}$. However, as $\mathrm{d}\vec{r} = \lim_{\Delta t \to 0} \Delta\vec{r}$, this means that the velocity $\vec{v} = \mathrm{d}\vec{r}/\mathrm{d}t$ is tangent to the path of motion at any given time. Since velocity is the derivative of position with respect to time, the velocity is always tangent to the path of motion.

For acceleration, we want to find $\Delta\vec{v} = \vec{v}' - \vec{v}$. Thus, the direction of acceleration is the same as that of $\mathrm{d}\vec{v}$. Acceleration is not tangent to the path of motion unless we have the special case of rectilinear motion.

§4.2 Rectangular Components of Velocity and Acceleration

We can specify the position of a vector $\vec{r}$ in terms of the unit vectors as

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Since velocity is the derivative of position, we can derive an expression for velocity from x, y, and z components,

$$\begin{aligned}\vec{v} &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}x}{\mathrm{d}t}\vec{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\vec{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\vec{k} \\ &= v_x\vec{i} + v_y\vec{j} + v_z\vec{k}\end{aligned}$$

Here, v_x, v_y, and v_z denote the velocity in the x, y, and z components respectively. The magnitude of velocity is speed, which is given as

$$v = \frac{\mathrm{d}s}{\mathrm{d}t},$$

where the speed v is obtained by finding the length s of the arc described by the particle and differentiating with respect to time t. Like velocity, acceleration can be defined in terms of a_x, a_y, and a_z along the x, y, and z axes is given by,

$$\begin{aligned}\vec{a} &= \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}v_x}{\mathrm{d}t}\vec{i} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\vec{j} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\vec{k} \\ &= a_x\vec{i} + a_y\vec{j} + a_z\vec{k}\end{aligned}$$

Therefore, the components of velocity and acceleration can be obtained by considering the motion in x, y, and z directions individually.

§5 September 20, 2017

§5.1 Curvilinear Motion Examples

Example 5.1 (Projectile Problem)

A particle initially moves with velocities $v_y(0)$ and $v_x(0)$. Determine the equations that describe its motion.

Solution. First, we establish a frame of reference taking the initial location of the particle to be the origin. Since we do not consider air resistance, this means that there is no force acting on the particle in the horizontal direction, and so there is no acceleration in this direction. In the vertical direction, the acceleration is equal to the acceleration due to gravity. Because the particle moves within the plane, the acceleration in the z direction is zero.

From the initial conditions, we know that that $t = 0$, $x = 0$, $y = 0$, $v_x(0) \neq 0$, and $v_y(0) \neq 0$. In the x direction, we have uniform rectilinear motion. Thus, $v_x = v_x(0)$ and $x = x_0 + v_x(0)t = v_x(0)t$. In the y direction, we have uniformly accelerated rectilinear motion, so $v_y = v_y(0) - gt$, and $y = v_y(0)t - gt^2/2$. Note that this results in rectilinear motion in orthogonal directions. the parametric equations $x(t)$ ad $y(t)$ describe a parabola. ■

Example 5.2

A basketball player shoots when she is 5 m from the backboard. The basket is 3.048 m from the ground. Knowing that the ball has an initial velocity $v(0)$ at an angle of 30° with the horizontal, determine the value of $v(0)$ when the basket is 228 mm from the backboard. Let $y(0) = 2$ m.

Solution. We need to find $v(0)$ such that $x = 5 - d$ when $y = 3.048$. Recall that for uniform rectilinear motion on the x direction, $x = x_0 + v_x(0)t$. Thus, since $x_0 = 0$, we obtain $x = v(0)\cos(30)t$ and therefore $t = x/v(0)\cos(30)$. In the y direction, we have $y = y(0) + v_y(0)t + \frac{at^2}{2} = y(0) + v(0)\sin(30)t - \frac{gt^2}{2} = y(0) + x\tan(30) - \frac{gt^2}{2}$. Thus, when $d = 0.228$ m, we find that $x = 4.77$ and $y = 3.048$. Therefore, substituting values gives $t = 0.590$ s and $v(0) = 9.34$ m/s at 30°. ■

§6 September 22, 2017

§6.1 Motion Relative to a Frame in Translation

For a moving frame of reference, any new position is given by

$$\vec{P} = P_x\vec{i} + P_y\vec{j} + P_z\vec{k}.$$

The rate of change of a vector is therefore

$$\frac{\mathrm{d}\vec{P}}{\mathrm{d}t} = \frac{\mathrm{d}P_x}{\mathrm{d}t}\vec{i} + \frac{\mathrm{d}P_y}{\mathrm{d}t}\vec{j} + \frac{\mathrm{d}P_z}{\mathrm{d}t}\vec{k}.$$

Suppose that we would like to determine the velocity and acceleration of a body in a frame that is in motion relative to another frame of reference. The absolute motion of a particle B is the motion of the frame plus the relative motion of B with respect to the moving frame. Let us call the origin of the moving frame A. Let $\vec{r}_B$ be the absolute position vector of point B that moves with the moving frame, $\vec{r}_A$ be the absolute position of the origin of the moving frame, and let $\vec{r}_{B/A}$ be the relative position of B with respect to A. We can then apply the triangle law for vectors to find that

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A},$$

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A},$$

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}.$$

Thus, we can find the absolute motion of a particle B by obtaining the motion of particle A, and the relative motion of B with respect to the frame attached in A that is in *translation*.

Example 6.1

When a small boat travels north at 3 miles/hour, a flag mounted on its stern forms an angle $\theta = 50°$ with the centre line of the boat. A short time later, when the boat travels east at 12 miles/hour, the angle once again 50°. Determine the speed and the direction of the wind.

Solution. Let w denote values pertaining to the wind, and b denote values pertaining to the boat. $\vec{v}_w$ and $\vec{v}_{b,1}$ are absolute to the fixed frame of the earth. Thus, we find that the flag is $\vec{v}_{flag,1} = \vec{v}_w - \vec{v}_{b,1} = \vec{v}_{w/b,1}$. We can rearrange this last equation to solve for $\vec{v}_w$. At this point, we do not know $\vec{v}_{w/b,1}$ or $\vec{v}_w$, but we do know $\vec{v}_{b,1}$ and the angle of the flag.

At the second position when the boat travels east, we again know the angle formed by the boat and the flag, and the velocity of the boat $\vec{v}_{b,2}$. We assume that the velocity of the wind remains the same. Here, we still do not know $\vec{v}_w$, and we also do not know $\vec{v}_{w/b,2}$.

We can now use the triangle vector law to solve for the unknowns, given that $\vec{v}_w$ remains the same for both positions. It may be advantageous to resolve into x and y components. Doing so, we find that $(\vec{v}_w)_x = 8.519$ miles/hour and $(\vec{v}_w)_y = -4.148$ miles/hour. Therefore, $\vec{v}_w = 9.475$ miles/hour at an angle of 25.96° ■

Example 6.2

The velocities of commuter trains A and B such that A is traveling west at 80 km/h and B is traveling 60 km/h at an angle of 25°. Knowing that the speed of each train is constant and that B reaches the crossing 10 minutes after A passed through the same crossing, determine the relative velocity of B with respect to A, and the distance between the fronts of the engines 3 min after A passed through the crossing.

Solution. From the geometry, we can apply the triangle vector law to determine the relative velocity of B with respect to A. That is, $\vec{v}_{b/a} = \vec{v}_b - \vec{v}_a$. Resolving into components and solving the equation, we find that $\vec{v}_{b/a} = 37.99$ m/s at 10.69°.

We recall that for rectilinear motion, the displacement is simply $d = vt$. The position of A is given as $vt = 80t$ km supposing that the intersection is arbitrarily denoted the origin. The position of B is therefore $d_{B,0}+$ ■

§7 September 25, 2017

§7.1 Motion of a Projectile

A *projectile* is any object upon which the only force acting is gravity. We assume that the initial position of the object is (x_0, y_0), the initial velocity of the object is $\vec{v}_0$, and there is no air resistance.

Example 7.1

A golf ball is hit at A with a speed of $v_A = 40$ m/s directed at an angle of 30° to the horizontal. Determine the distance d that the ball strikes B, which lies on an incline with a slope of 0.2.

Solution. We are given $v_A = 40$ m/s, and that $\theta = 30°$. We denote the horizontal length and the height at which the ball makes contact at B. Thus, we first note that the angle of the incline is $\tan^{-1}(0.2) = 11.31°$. We can then express the horizontal length as $R = d\cos(11.31)$ and $h = d\cos(11.31)$. We can now find the horizontal component $R = v_x(0)t = v_A\cos(\theta)t$ and the vertical component $h = v_y(0)t - gt^2/2 = v_A\sin(\theta)t - gt^2/2$. Combining these equations, we find that $d = 94.1$ m. ■

§7.2 Normal and Tangential Components of Curvilinear Motion

Let $s(t)$ denote the path function of a particle with respect to time. We will now utilize a coordinate system that uses normal N and tangential T components of motion. That is, these axes are normal and tangential to the path of motion, and have their origin located on the particle at all times. We will refer to the unit normal vector and tangential vector as $\vec{u}_n$ and $\vec{u}_t$ respectively.

We can consider the paths as being constructed by a series of differential arc segments. Each segment $\mathrm{d}s$ is the arc length of an associated circle of radius of curvature ρ and of centre O'. By convention, T is positive in the direction of increasing s, and N is positive in the direction towards the centre of curvature O'.

The particle's velocity is always tangent to the path of the particle, and has a magnitude equal to the slope $v = \mathrm{d}s/\mathrm{d}t$. Acceleration on the other hand, is the rate of change of the velocity vector. Therefore, given unit direction vectors $\vec{u}_t$ and $\vec{u}_n$, we find that

$$\vec{v} = v\vec{u}_t,$$

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t}\vec{u}_t + \frac{v^2}{\rho}\vec{u}_n,$$

where $a_t = \mathrm{d}v/\mathrm{d}t$ is the tangential component of acceleration and $a_n = v^2/\rho$ is the normal component of acceleration. From this, we conclude that the tangential component reflects the change in speed of the particle, while the normal component reflects the change in direction of the particle. There are times when the radius of curvature is not obvious. In the cases when the path is expressed as $y = f(x)$, then the radius is

$$\rho = \frac{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}}{\left\|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right\|}.$$

§8 September 27, 2017

§8.1 Tangential and Normal Component Examples

Example 8.1

The car starts at $s = 0$ and increases its speed at $a_t = 4$ m/s^2 on a track of radius 40 m. Determine the time when the magnitude of acceleration becomes 20 m/s^2. At what position s does this occur.

Solution. We are given that $s(0) = 0$, $v(0) = 0$, $a_t = 4$, and $\rho = 40$. We want to determine the time t and position s when $\|\vec{a}\| = 20$.

The components of acceleration in this case are $a_t = 4$ and $a_n = v^2/40$. Now, since $\|\vec{a}\| = \sqrt{(a_t)^2 + (a_n)^2}$, we solve to find that $v = 28$ m/s. In the tangential direction, $v = v_0 + a_t t$. We can then solve this with $v = 28$, $a_t = 4$, and $v(0) = 0$. Solving this, we obtain $t = 7$ s. Now, we can find s by substituting known values into $v^2 = v(0)^2 + 2a_t(s - s(0))$ to find $s = 98$ m. ∎

Example 8.2

If a roller coaster starts from test at A and its speed increases at $a_t = (6 - 0.06s)$ m/s^2, determine the magnitude of its acceleration when it reaches B where $s(B) = 40$ m. B is located 30 m to the right of the origin, while A is at a location given by $y = x^2/100$.

Solution. We want to determine $\|\vec{a}\|$ when $s(B) = 40$. Substituting $s = 40$, we find that $a_t = 6 - 0.06(40) = 3.6$ m/s^2. We can find v by recalling that $a_t \mathrm{d}s = v\mathrm{d}v$. Integrating both sides, we find

$$\begin{aligned} a_t \mathrm{d}s &= v \mathrm{d}v \\ \int_0^s (6 - 0.06s)\mathrm{d}s &= \int_0^v v \mathrm{d}v \\ 6s - 0.03s^2 &= \frac{v^2}{2} \end{aligned}$$

Solving for v when $s = 40$, we find that $v = 19.6$ m/s^2. Now, we find the radius of curvature

$$\begin{aligned} \rho &= \frac{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}}{\left\|\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right\|} \\ &= \frac{\left(1 + \left(\frac{x}{50}\right)^2\right)^{3/2}}{\frac{1}{50}} \\ &= 50\left(1 + \left(\frac{30}{50}\right)^2\right)^{3/2} \\ &= 79.3 \end{aligned}$$

Now, we note that $a_n = 19.6^2/79.3 = 4.84$ m/s^2. Therefore, $\|\vec{a}\| = \sqrt{3.6^2 + 4.84^2} = 6.03$ m/s^2. ■

§9 September 29, 2017

§9.1 Review of Friction

We recall that *friction* resists relative motion or sliding between two surfaces. When a force P is applied, the magnitude of the friction force F increases until it reaches a maximum value of F_s. When the load force P is increased further, the friction force can no longer balance, and so the object starts sliding. When the object is in motion, the magnitude of the friction force decreases to F_k. *Static friction* and *kinetic friction* are given as

$$\begin{aligned} F_s &= \mu_s N, \\ F_k &= \mu_k N. \end{aligned}$$

When a rigid body comes into contact with a surface, four scenarios may result:

1. The forces applied to the body do not move it along the surface of contact, so there is no friction force. This occurs for instance, when all forces and reactions are directed vertically, so there is no tendency to slide across the surface.

2. The forces applied to the body tend to move the body along the surface, but are not large enough to initiate motion. The static friction force F can be found from equilibrium equations. But since F has not reached its maximum value, $F_s = \mu_s N$ cannot be used.

3. The forces applied to the body are such that motion is about to occur. In this case, the friction force has reached its maximum value F_s, in the direction that opposes the impending motion. Thus, the relevant equation and equilibrium equations can be used.

4. The forces applied to the body are used to slide the body along the surface. The equilibrium equations cannot be applied, the friction is now given by $F_l = \mu_k N$ in the direction that opposes the motion.

The *angle of static friction* ϕ_s denotes the angle formed by the resultant of the normal and friction force to the normal at the maximum F_s before motion occurs. When motion actually occurs, the magnitude of the friction force drops to F_k, and so the *angle of kinetic friction* ϕ_k is now used. The results are summarized below.

$$\tan\phi_s = \frac{F_m}{N} = \frac{\mu_s N}{N} = \mu_s,$$

$$\tan\phi_k = \frac{F_k}{N} = \frac{\mu_k N}{N} = \mu_k.$$

Example 9.1

A 40 kg package is at rest on an incline of 20° when a force P is applied to it at an angle of $-30°$. Determine the magnitude of P if 4s is required for the package to travel 10m up the include. The static and kinetic coefficients of friction between the package and the incline are 0.30 and 0.25 respectively.

Solution. We are given $m = 40\text{kg}$, $\mu_s = 0.30$, $\mu_k = 0.25$. The angle of friction $\phi_s = \tan^{-1}(0.3) = 16.7° < 20°$. If $P = 0$, then the block moves downwards, since $\phi_s < 20°$. From kinematics, we know that acceleration is a constant, since the forces are constant. With $x_0 = 0$ and $v_0 = 0$, we know that $x = at^2/2$. However, when $t = 4$, we know that $x = 10$, so solving for acceleration, we find that $a = 1.25\ \text{m/s}^2$ at an angle of 20°.

We now denote the y direction as normal to the block on the incline, with x the direction up the incline. Summing the forces in the y direction, we have $N - P\sin(50°) - mg\cos(20°) = 0$, where N is the normal force. Now, applying Newton's Second Law in the x direction, we have $P\cos(50°) - mg\sin(20°) - \mu_k N = ma$. Simultaneously solving these two equations by substituting N, we find that $P = 612.5\text{N}$. ■

§9.2 Translating Axes Example

Example 9.2

Car B is traveling along a curved road with a speed of 15m/s while decreasing its speed by 2m/s^2. At this same instant, car C is traveling along the straight road with a speed of 30m/s while decelerating at 3m/s^2. Determine the velocity and acceleration of car B relative to car C. At the start, B is located at 210° from C. The radius of curvature of the path of B is 100m.

Solution. First, we need to set up a set of reference axes. Now, we note that $\vec{v}_C = \left(-30\vec{j}\right)\text{m/s}$ and $\vec{v}_B = \left(15\cos(60)\vec{i} - 15\sin(60)\vec{j}\right)\text{m/s}$. From the triangle law of vectors, we can find the relative velocity $\vec{v}_{B/C} = \vec{v}_B - \vec{v}_C = \left(7.5\vec{i} + 17.01\vec{j}\right)\text{m/s}$. The magnitude of the relative velocity is therefore $\left\|\vec{v}_{B/A}\right\| = \sqrt{7.5^2 + 17.01^2} = 18.6\text{m/s}$. The angle is therefore $\theta = \tan^{-1}(17.01/7.5) = 66.2°$.

Now for acceleration, we note that acceleration is -3m/s for C, so $\vec{a}_C = -3\vec{j}\text{m/s}^2$. The tangential component of acceleration of B can be expressed as $(\vec{a}_B)_t = -2\cos(60)\vec{i} + 2\sin(60)\vec{j} = \left(-\vec{i} + 1.732\vec{j}\right)\text{m/s}^2$. The normal component is $(a_B)_n = v^2/\rho = 15^2/100 = 2.25\text{m/s}^2$. We can then resolve this into $(\vec{a}_B)_n = 2.25\cos(30)\vec{i} + 2.25\sin(30)\vec{j} = 1.948\vec{i} + 1.125\vec{j}$. Adding both components of acceleration for B, we obtain $\vec{a}_B = (1.948 - 1)\vec{i} + (1.125 + 1.732)\vec{j}$. Thus, subtracting the acceleration of C, we find $\vec{a}_{B/C} = \left(0.948\vec{i} - 0.143\vec{j}\right)\text{m/s}^2$. ■

§10 October 2, 2017

§10.1 Newton's Second Law of Motion

Newton's Second Law states that when an unbalanced force acts on a particle, the particle will acceleration in the direction of the force with a magnitude that is proportional to that force. The constant of proportionality is the mass of the particle. The mass is a quantitative measurement of an object's inertia. Stated mathematically, this states that

$$\sum \vec{F} = m\vec{a},$$

where $\vec{F}$ is the net force, $\vec{a}$ is the acceleration with the same direction as $\vec{F}$, and m is the mass. Note also that $\vec{a}$ must be obtained in a fixed frame of reference. That is, it is the absolute acceleration.

Note that we can use the rectangular component form by splitting the acceleration and force n each of the x, y, and z directions. We consider a particle moving relative to an inertial (fixed) x, y, and x coordinate system. Thus, forces F_x, F_y, and F_z act upon the particle, so we can separate Newton's Second Law into components.

$$\sum F_x = ma_x,$$

$$\sum F_y = ma_y,$$

$$\sum F_z = ma_z.$$

We can also consider tangential and normal components.

$$\sum F_t = m\frac{\text{d}v}{\text{d}t},$$

$$\sum F_n = m\frac{v^2}{\rho}.$$

Remark 10.1. The second law of motion was originally stated by Newton as

$$\vec{L} = m\vec{v},$$

where $\vec{L}$ is the *linear momentum* of a particle. The direction is the same as the velocity of the particle, while the magnitude is equal to the product of the mass and the speed. Thus, the second law can also be expressed as the rate of change of linear momentum.

The four primary dimensions are force, mass, length, and time. We can define three of the four, with the fourth dimension being derived. In SI units, the base units of length m, mass kg, and time s are used to derive the units of force $N = kg \cdot m/s^2$. In US units, the base units of force lb, length ft, and time s are used to derive the units of mass $slug = lb \cdot s^2/ft$. In US units, the acceleration due to gravity is approximately 32.2 ft/s^2.

§11 October 4, 2017

§11.1 Equations of Motion

When more than one force is acting on a particle, it is the resultant force obtained by vector summation that is of interest. Note that if the resultant force is zero, then there is no change is the motion. This is not necessarily implying that there is no motion. In a free body diagram, the object is shown free from all of its surroundings, supports, and joints. All of the external forces are shown on the FBD. *Free Body Kinetic Diagrams* (or *kinetic diagrams*) on the other hand, simply show $m\vec{a}$ instead of $\vec{F}$.

Consider a particle of mass m that is acted upon by several forces. Recall that Newton's Second Law states that

$$\sum \vec{F} = m\vec{a}.$$

In a FBD, we are concerned with the left side of the above equation. Thus, we define the following.

- Body: Define the system by isolating the body to be considered. If the bodies can be separated into individual components, multiple FBD and KD may be drawn.

- Axes: Define an appropriate coordinate system. This may be Cartesian, normal and tangential, or radial and transverse.

- Reaction Forces: Replace constraints and supports with the forces that they produce on the body being considered.

- Applied Force: Draw applied forces and any body (field) forces on the diagram. This may include weight, magnetic forces, or pulling forces.

- Dimensions: Include angles and distances that are important for solving the problem.

In a KD, we are concerned with the right side of the above equation. In statics problems, we are concerned with bodies in equilibrium, so the inertial term in Newton's Second Law is zero. For dynamics problems however, this is not the case. Thus, we define the following.

- Body: This is defined in the same manner as the one defined for the FBD. We draw the body for the KD beside the body for the FBD.

- Inertial Term: Draw the $m\vec{a}$ term consistent with the coordinate system defined previously. This term is usually separated into components (ma_x and ma_y, or ma_n and ma_t). When these values are not known, they are drawn arbitrarily in the positive direction.

Example 11.1

A 400kg mine car is hoisted up an incline of slope 8/15 using a cable and motor M. For a short time, the forces on the cable are $F = (3200t^2)$N, where t is in seconds. If the car has an initial velocity of $v(0) = 2$m/s at $s = 0$ and $t = 0$, determine the distance that the mine cart has moved up the incline when $t = 2$s.

Solution. We first separate the force into vertical and horizontal components, assuming that there is no friction. We now consider the FBD, with weight W acting directly downwards, and the normal force N acting normal to the surface of the incline. Select the x and y components to be along the incline and the normal. Thus, we resolve the x and y components.

$$\begin{aligned} \sum F_x &= ma_x \\ 3200t^2 - 400(9.81)\left(\frac{8}{17}\right) &= 400a_x \\ a_x &= 8t^2 - 4.616 \end{aligned}$$

Since we are asked to find the displacement s at $t = 2$, we integrate acceleration twice to obtain an expression for displacement. Therefore, $v_x = 8t^3/3 - 4.616t + 2$, where the 2 comes from the initial velocity. Similarly, $s_x = 2t^4/3 - 4.616t^2/2 + 2t$. Evaluating this expression at $t = 2$, we find that $s_x = 5.43m$ up the incline. ■

§12 October 6, 2017

§12.1 Angular Momentum

Let P be a particle with mass m moving with respect to a newtonian frame of reference $Oxyz$. Recall that $m\vec{v}$ is the linear momentum of a particle. We define the *moment of momentum* as

$$\vec{H}_O = \vec{r} \times m\vec{v},$$

where $\vec{H}_O$ is the *angular momentum* (also known as moment of momentum), and $\vec{r}$ is the position vector of the particle P extending from the origin with linear momentum $m\vec{v}$. Note that $\vec{H}_O$ is perpendicular to the plane containing $\vec{r}$ and $m\vec{v}$, and that its magnitude is

$$\left\|\vec{H}_O\right\| = rmv \sin\phi,$$

where ϕ is the angle between $\vec{r}$ and $m\vec{v}$. The components of angular momentum can be resolved into the coordinate axes.

$$\begin{aligned} H_x &= m(yv_z - zv_y), \\ H_y &= m(zv_x - xv_z), \\ H_z &= m(xv_y - yv_x). \end{aligned}$$

When the particle is moving along the xy plane only, then $z = v_y = 0$ in the above equations.

Differentiating angular momentum, we find that the sum of the moments about O of the forces acting on the particle is equal to the rate of change of angular momentum of the particle about O.

$$\sum \vec{M}_O = \vec{r} \times m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{r} \times m\vec{a} = \vec{r} \times \sum \vec{F}.$$

§13 October 16, 2017

§13.1

Example 13.1

A box has a mass m and slides down a chute with the shape of a parabola. If it has an initial velocity v_0 at the origin, determine the velocity as a function of x. Also determine the normal force on the box and the tangential acceleration as a function of x. The curve is $y = -0.5x^2$.

Solution. First, we draw the free body diagram with weight W acting downwards, the normal force N acting normal to the surface at a distance of x, and the tangential force acting forwards along the curve. Summing the forces in the normal direction and setting this equal to the mass multiplied by the normal component of acceleration, we obtain

$$W\cos(\theta) - N = m\frac{v^2}{\rho}.$$

We can obtain the radius of curvature, since

$$\begin{aligned}\rho &= \frac{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}}{\left\|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right\|}\\ &= \frac{\left(1+x^2\right)^{3/2}}{\|1\|}\\ &= \left(1+x^2\right)^{3/2}\end{aligned}$$

Because $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$, we can rearrange terms to find that $\mathrm{d}s = \sqrt{1+(\mathrm{d}y/\mathrm{d}x)^2}\mathrm{d}x = \sqrt{1+x^2}\mathrm{d}x$. Because $\cos(\theta) = dx/ds$, we find that it can be expressed as $1/\sqrt{1+x^2}$. From the first equation above, this means that

$$N = \frac{mg}{\sqrt{1+x^2}} - \frac{mv^2}{(1+x^2)^{3/2}}.$$

We now sum the forces in the tangential direction to be equal to mass multiplied by the tangential acceleration. This gives us

$$W\sin(\theta) = ma_t.$$

But we know that $\tan(\theta) = \mathrm{d}y/\mathrm{d}x$, so $\sin(\theta) = \cos(\theta)\mathrm{d}y/\mathrm{d}x = x/\sqrt{1+x^2}$. Substituting this into the above equation, we obtain

$$mg\left(\frac{-x}{\sqrt{1+x^2}}\right) = ma_t.$$

Now that we have the tangential acceleration, we can obtain velocity through

$$\begin{aligned}v\mathrm{d}v &= a_t\mathrm{d}s\\ v\mathrm{d}v &= g\left(\frac{x}{\sqrt{1+x^2}}\right)\sqrt{1+x^2}\mathrm{d}x\\ \int_{v_0}^{v} v\mathrm{d}v &= \int_0^x gx\mathrm{d}x\\ \frac{v^2}{2} - \frac{v_0^2}{2} &= \frac{gx^2}{2}.\end{aligned}$$

Now, we can now solve this for v in the expression for N to obtain

$$N = \frac{m}{\sqrt{1+x^2}}\left(g - \frac{v_0^2 + gx^2}{1+x^2}\right).$$

■

§14 October 18, 2017

§14.1 Work and Energy Methods

We will explore a useful method for solving problems involving force, displacement, and velocity. This method is termed work and energy methods. Consider a force $\vec{F}$ that will do work on a particle only when the particle undergoes a displacement in the direction of the force. It is only the displacement that is of interest. If the force causes the particle to move along the path s from $\vec{r}$ to $\vec{r}'$, then the displacement is $\mathrm{d}\vec{r} = \vec{r}' - \vec{r}$. The magnitude of $\mathrm{d}\vec{r}$ can be approximated by $\mathrm{d}s$. If the angle between the tails of $\vec{F}$ and $\mathrm{d}\vec{r}'$ is θ, then the work done by $\vec{F}$ can be expressed as

$$\mathrm{d}U = F\cos(\theta)d\theta,$$

$$\mathrm{d}U = \vec{F}\cdot\mathrm{d}\vec{r}.$$

§14.2 Work of a Variable Force

In the most general terms, the work of a force on a particle that undergoes a finite displacement from $\vec{r}_1$ to $\vec{r}_2$ is given as

$$U_{1,2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot\mathrm{d}\vec{r} = \int_{s_1}^{s_2} F\cos(\theta)d\theta.$$

Note the special case where the force is a constant.

§15 October 23, 2017

§15.1 Momentum and Impulse Examples

Example 15.1

A 4lb sphere A is connected to a fixed point O by an inextensible cord of length 3.6ft. The sphere is resting on a frictionless horizontal surface at a distance of 1.5ft from O when it is given a velocity v_0 in a direction perpendicular to line OA. It moves freely until it reaches position A', when the cord becomes taut when it is extended 3.6ft. Determine the maximum allowable velocity v_0 if the impulse of the force exerted on the cord is not to exceed 0.8lbs.

Solution. Applying the principle of impulse and momentum, we find that $mv_0\cos(90-\theta) - F\Delta t = 0$. We are given the condition that $F\Delta t \leq 0.8$lbs. From trigonometry, this is the same as saying that $mv_0\sin(\theta) \leq 0.8$. Thus, $v_0 \leq 0.8/m\sin(\theta) = 7.084$ft/s. ■

§15.2 Central Impact

The *line of impact* is the common normal to the contact surface. *Central impact* occurs when the two mass centers lie on the line of impact, while *eccentric impact* occurs otherwise. In this course, we are concerned only with central impact. Central impact can be separated into *direct central impact* where the velocity of both particles are directed along the line of impact, and *oblique central impact* where the velocity of at least one particle is not collinear with the line of impact.

§15.3 Direct Central Impact

During direct central impact, we first start with velocities v_A and v_B of particles A and B respectively before impact. When impact occurs, we separate the process into the duration when the particles deform, then recover from this deformation from the impact. During the *period of deformation*, deformation occurs from zero to the maximum. During the *period of restitution*, deformation can be fully recovered (*elastic*), permanent (*plastic*), or partially recovered. After the impact, we have v'_A and v'_B.

When there are no forces in a particular direction, we have conservation of momentum in this direction. This can be expressed mathematically as,

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B.$$

During the period of deformation, we consider the first position to be just before impact, and the second position to be at maximum deformation. Applying the principle of impulse and momentum, we can equate the results from the two positions to obtain,

$$m_A v_A - \int P(t)\mathrm{d}t = m_A u.$$

During the period of restitution, we consider the first position to be at maximum deformation, and the second position to be from just having recovered from elastic deformation. Thus, we obtain,

$$m_A u - \int R(t)\mathrm{d}t = m_A v'_A.$$

To simplify calculations, we define the *coefficient of restitution*, defined as

$$e = \frac{\int R(t)\mathrm{d}t}{\int P(t)\mathrm{d}t} = \frac{u - v'_A}{v_A - u} = \frac{v'_B - u}{u - v_B} = \frac{v'_B - v'_A}{v_A - v_B},$$

where $0 \leq e \leq 1$.

§16 October 30, 2017

§16.1 Translation and Fixed Axis Rotation

Consider a rigid body in *translation*. By definition, the direction of any straight line in the body does not change in length or direction. Thus, letting A and B be two particles inside the body, we note that the relative displacement from A to B $\vec{r}_{B/A}$ is constant throughout the translation. Thus, its derivative is zero. Thus, we can differentiate $\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$ to obtain

$$\vec{v}_B = \vec{v}_A,$$

$$\vec{a}_B = \vec{a}_A.$$

Thus, when a rigid body is in translation, all points of the body have the same velocity and the same acceleration at any given instant. In the case of curvilinear translation, the velocity and acceleration change in direction as well as magnitude at every instant. In the case of rectilinear translation, all particles of the body move along parallel straight lines, with the velocity and acceleration maintaining the same direction at all times. In translation, we can treat the rigid body as a particle since all of the particles move with the same velocity and acceleration. In other words, there is no relative motion between any two points inside the body.

Consider a rigid body in *rotation*. By definition, the rigid body rotates about a fixed axis AA'. Let point P be a point of the body and $\vec{r}$ be its position vector with respect to a fixed frame. The angular velocity and acceleration are

$$\vec{\omega} = \omega\vec{k} = \theta\vec{k},$$

$$\vec{\alpha} = \alpha\vec{k} = \omega'\vec{k} = \theta''\vec{k},$$

where k is the unit vector along the AA' axis. Now considering linear velocity and linear acceleration, we obtain

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{\omega} \times \vec{r},$$

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}).$$

where r is the position vector. Note that the linear velocity $\vec{v}$ is tangential to the path of motion. However, we can express the rotation of a rigid body about a fixed axis by examining the motion of a representative slab in a reference plane perpendicular to the axis of rotation. Choosing the z axis as the rotation axis and the xy plane as the reference plane perpendicular to it, we obtain a *representative slab*. Let $\vec{u}_t$ be the tangential unit vector pointing counterclockwise, and $\vec{u}_n$ be the normal unit vector pointing towards the the center of rotation. Thus, we obtain the following expressions for velocity and acceleration

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega\vec{k} \times \vec{r} = r\omega\vec{u}_t,$$

$$\vec{a} = \alpha\vec{k} \times \vec{r} - \omega^2\vec{r} = (r\alpha)\,\vec{u}_t + \left(r\omega^2\right)\vec{u}_n,$$

where r is the radius, ω is the magnitude of angular velocity, the magnitude of the tangential component of acceleration is $a_t = r\alpha$ and the magnitude of the normal component of acceleration is $a_n = r\omega^2$.

§16.2 Equations of Rotation

The motion of a rigid body rotating about a fixed axis AA' is said to be known when we can express its angular coordinate θ as a known function of t. In practice however, we can seldom describe the rotation of a rigid body by a relation between θ and t. More often, the conditions of motion are specified by the angular acceleration of the body. Thus, we obtain

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t},$$

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta}.$$

This is analogous to the equations for linear velocity and acceleration. We obtain the following special cases of rotation,

1. *Uniform Rotation*: This case is characterized by the fact that the angular acceleration is zero, so $\alpha = 0$. Thus, the angular velocity is constant and the angular position is given by
$$\theta = \theta_0 + \omega t.$$

2. *Uniformly Accelerated Rotation*: In this case, the angular acceleration is constant, so $\alpha = C$ for some constant C. We can derive the following formulas relating angular velocity, angular position, and time in a manner similar to those for linear acceleration.
$$\omega - \omega_0 = \alpha t,$$
$$\theta - \theta_0 = \omega_0 t + \frac{\alpha t^2}{2},$$
$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0).$$

Example 16.1

Two friction wheels A and B are both rotating freely at 300rpm counterclockwise when they are brought into contact. After 12s of slippage during which both wheels have a constant angular acceleration, wheel B reaches a final angular velocity of 75rpm counterclockwise. Determine the angular acceleration of each wheel during the period of slippage, and the time at which the angular velocity of wheel A is equal to zero. The radius of A is 2.5in, and the radius of B is 3in.

Solution. Note that after 12 seconds, we have uniform rotation so $\alpha_A = \alpha_B = 0$. $\omega = 2\pi(75/60) = 7.85\text{rad/s}$ and $\omega_0 = 2\pi(300/60) = 31.41\text{rad/s}$. Substituting $t = 12$ in the equation $\omega - \omega_0 = \alpha t$, we find that $\alpha = -1.9635\text{rad/s}$. We also need to determine the angular velocity of A at twelve seconds. ■

General motion describes motion that is not translation or rotation about a fixed axis. The two special cases of translation and rotation may be combined to produce general motion.

§17 November 6, 2017

Example 17.1

Rod AB of length 20in can slide freely along the floor and inclined plane. The right side B is located 12in above the ground on an incline that has a slope of 12/5. The left end A is on the flat ground. At the instant shown, the velocity of end A is 4.2ft/s to the left. Determine the angular velocity of the rod, and the velocity of end B of the rod.

Solution. Note that the angle of the incline is $\theta = \tan^{-1}(12/5) = 67.38°$, and the angle of AB with the ground is $\beta = \sin^{-1}(12/20) = 36.87°$. We need to find ω_{AB} and $\vec{v}_B$. We now want to determine the relations between the unknowns and the given velocities. Although we do not know the magnitude of $\vec{v}_B$, we do know the direction is down the incline. We also know that the relative velocity $\vec{v}_{B/A}$ must be normal to AB. Thus, we can draw a velocity triangle with all angles known and with only the velocity of A known.

We then can simply apply the law of sines to obtain $\vec{v}_{B/A}$ and $\vec{v}_B$. In this case, we have $\vec{v}_{B/A}/\sin(\theta) = \vec{v}_A/\sin(\phi)$, where $\phi = 59.59°$. Substituting known values, we obtain

$\vec{v}_{B/A} = 4.5$ft/s. To find the angular velocity, we have $\vec{v}_{B/A} = \overline{AB}\omega_{AB} = \omega_{AB}(20/12)$, so $\omega_{AB} = 2.7$rad/s. Note that we divided by 12 to convert to feet. Applying the law of sines, we can also determine $\vec{v}_B = 3.9$ft/s at $67.38°$ south from west. ■

Example 17.2

The 80mm radius wheel rolls (without sliding) to the left with a velocity of 900mm/s. The wheel is attached to a 250mm rod at point A, where the line drawn from the center of the wheel to the connection point forms an angle β east from south. The other end of the 250mm rod is attached to collar B that is $160mm$ from the ground. Knowing that the distance AD is 50mm from point A to the center, determine the velocity of the collar and the angular velocity of rod AB when $\beta = 0$ and when $\beta = 90$.

Solution. Note that we are given $\vec{v}_D = 900$mm/s to the left, and the radius of the wheel is $r = 80$mm. We want to find $\vec{v}_B$ and ω_{AB}. We first consider rod AB. Note that A is the point where the rod is connected to the wheel, and D is the center of the wheel. Let point C denote the point of contact of the wheel with the ground. We would like to determine the velocity of point A.

First we solve for $\beta = 0$. Now, consider wheel D. Since we are not sliding, we have the relative velocity of point C to the other surface is 0. But since the ground is not moving, we have $\vec{v}_C = 0$. We want to determine the angular velocity of D. But $\vec{v}_D = \vec{v}_C + \vec{v}_{D/C}$. But since $\vec{v}_C = 0$, we can determine the relative velocity of D with respect to C. It is exactly equal to the velocity of D. Now, $\vec{v}_{D/C} = \overline{CD}\omega_{CD}$, where $\overline{CD}$ indicates the distance between D and C. But the length of CD is just the radius, so the angular velocity is $\omega_D = 900/80 = 11.25$rad/s.

It remains to find $\vec{v}_B$. Since A in on the wheel, we can find $\vec{v}_A = \vec{v}_C + \vec{v}_{A/C}$. But since $\vec{v}_C = 0$, and since $\vec{v}_{A/C} = \overline{AC}\omega_D$, we have $\vec{v}_A = \vec{v}_{A/C} = \overline{AC}\omega_D = (80 - 50)(11.25) = 337.5$mm/s.

Now consider rod AB. Note then that we have $\vec{v}_A = 337.5$mm/s to the left, and that $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$. But $\vec{v}_{B/A}$ is normal to AB. Summing components in the vertical direction, we note that $\vec{v}_{B/A}$ must equal 0, so $\omega_{AB} = 0$ meaning that there is no rotation. Therefore, $\vec{v}_B = \vec{v}_A = 337.5$mm/s.

Finally, we solve for $\beta = 90$. The velocity of $\vec{v}_B$ remains horizontal. The rotation ω_D remains the same as before. First consider the wheel, where $\vec{v}_D = \vec{v}_{D/C} + \vec{v}_C$. But since the last term is zero, we can directly find $\vec{v}_{D/C}$, where this can be used to obtain $\omega_D = 11.25$rad/s. Now, we find $\vec{v}_A = \vec{v}_D + \vec{v}_{A/D} = -900\vec{\imath} + \left(\overline{AD}\omega_D\right)\vec{\jmath} = -900\vec{\imath} + 562.5\vec{\jmath}$.

Now, consider the other rigid body AB. We can determine that the angle that rod AB forms with the horizontal is $\sin(\alpha) = 80/250$, so $\alpha = 18.663°$. But $\vec{v}_{B/A}$ is perpendicular to the rod AB. Thus, we know the direction of $\vec{v}_{B/A}$, with its magnitude being $\overline{AB}\omega_{AB}$. We have unknowns of ω_{AB} and $\vec{v}_B$. But separating into vertical and horizontal components, we can determine both unknowns. Doing so, we obtain $\omega_{AB} = -2.375$rad/s and $\vec{v}_B = 710$mm/s. ■

§18 November 8, 2017

§18.1 Instantaneous Center of Rotation in Plane Motion

Consider the general plane motion of a rigid body. We will show that, at any given instant, the velocities of the various particles of the rigid body are the same as if the body were rotating about an axis perpendicular to the plane of the body, called the *instantaneous axis of rotation*. This axis intersects the plane of the rigid body at a point C, called the instantaneous center of rotation. At every instant in time, the rigid body appears to rotate about this point. This gives us an alternative method for solving problems involving the velocities of points on an object in plane motion.

Thus, to calculate the velocity at any point in the rigid body, we can simply connect an imaginary line from the instantaneous center of rotation to that point. The velocity at that point is then perpendicular to this imaginary line.

Example 18.1

Suppose that we have a wheel traveling with a velocity v to the left. Note that this can be considered the velocity of the center of the wheel. After a time increment Δt, the wheel has rotated a certain amount. Consider the two cases.

1. If we are stuck, then rotation occurs much more than the velocity travelled forwards.
$$v\Delta t < R\Delta\theta.$$
2. If we slip, then rotation occurs much less than the velocity travelled forwards.
$$v\Delta t > R\Delta\theta.$$
3. Under normal conditions, we have that rotation is directly related to the velocity travelled forwards.
$$v\Delta t = R\Delta\theta.$$
Rearranging this, we obtain
$$v = R\omega.$$

To find the instantaneous centre of rotation, we can obtain the velocities at two points within the rigid body. Drawing a perpendicular line from the direction of the velocity at that point, the intersection is necessarily the instantaneous centre of rotation. Note that this does not apply to acceleration, as the instantaneous centre of rotation applies only to velocity calculations.

Example 18.2

Knowing that at the instant shown the angular velocity of rod AB is 15rad/s clockwise, determine the angular velocity of BD, and the velocity of the midpoint of rod BD. A is attached to the top, with B located 200mm below it. D is located 600mm to the left and 250mm below B, and E which is fixed is located 200mm right of D.

Solution. The rotation of AB is about a fixed axis. Thus, $\vec{v}_B = 0.2(15) = 3$m/s clockwise, and is towards the left as it is normal to the instantaneous center of rotation. Now, DE is also rotated about a fixed axis, since E is fixed. So, we can conclude that $\vec{v}_D$ is vertical at this instant. Now, we can determine the instantaneous centre of rotation to be directly to the right of D and directly below B. Call this point C. Thus, $v_B = \overline{BC}\omega_{BD}$. Thus, with $v_B = 3$ and $\overline{BC} = 0.25$, we find that $\omega_{BD} = 12$rad/s.

Now, we find the velocity at the midpoint of BD. But note that the midpoint of BD which we call F, ensures that the lengths of DF and CF are the same. Now, we need to solve for $v_F = \overline{CF}\omega_{BD}$. But we know that the length of CF is half the length of BD, so $\overline{CF} = 0.325$m. Thus, we can substitute known values of $\overline{CF}$ and ω_{BD} to obtain $v_F = 0.325(12) = 3.9$m/s. The angle is determined from $\tan(\beta) = 0.250/.600 = 22.6$ degrees North from West. Putting this together, we obtain $\vec{v}_F = 3.9$m/s at $22.6°$ North from West.

■

§19 November 15, 2017

§19.1 Absolute and Relative Acceleration in Plane Motion

Recall that the relative acceleration of B with respect to A is given as $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$. But the relative acceleration term $\vec{a}_{B/A}$ is related to the rotation of B about A. Namely, it is comprised of a normal and tangential components. Thus, we obtain the following expression,

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_n + \left(\vec{a}_{B/A}\right)_t.$$

To obtain normal and tangential components of acceleration, velocity is required.

$$\left(a_{B/A}\right)_n = \overline{AB}\omega^2,$$

$$\left(a_{B/A}\right)_t = r\alpha = \overline{AB}\alpha,$$

where ω is the angular velocity and α is the angular acceleration. Note that $\left(\vec{a}_{B/A}\right)_n$ is along BA, whereas $\left(\vec{a}_{B/A}\right)_n$ is tangential to this line.

Example 19.1

Collar A is 4in from a block to the left. D is located 7.5in below this fixed block, and is part of rod ADB, where B extends 3in below D and a bit to the left (along the slope of AD). E is located directly to the right of D at a distance of 6in. This forms rod DE. Knowing that at the instant shown, the velocity of collar A is zero and its acceleration is 0.8ft/s^2 to the left, determine the angular acceleration of rod ADB, and the acceleration of point B.

Solution. We are given that $\vec{v}_A = 0$ and $\vec{a}_A = 0.8$ft/s^2 to the left. From the geometry, we can determine the length of $AD = 8.5$in and the length of $AB = 11.9$in. We can determine the instantaneous center of rotation I, since we know both the direction of $\vec{v}_D$ and $\vec{v}_A$. It is perpendicular to both velocity directions, so it is directly below A and directly to the right of D. Because the velocity of A towards I is 0, we have $0 = \vec{v}_A = \omega_{AB}\overline{AI}$, so $\omega_{AB} = 0$. Thus, $\vec{v}_D = \omega_{AB}\overline{DI} = 0$. Therefore, since $\vec{v}_D = 0$, we have $\omega_{DE} = 0$.

Now, rod DE rotates about E, so $\vec{a}_D = (\vec{a}_D)_n + (\vec{a}_D)_t$. But since $(\vec{a}_D)_n = \overline{DE}\left(\omega_{DE}\right)^2$, this evaluates to zero. Thus, we have $\vec{a}_D = \overline{DE}\alpha_{DE} = (0.5\text{ft})\alpha_{DE}$. Now considering rod AB, we have $\vec{a}_D = \vec{a}_A + \left(\vec{a}_{D/A}\right)_n + \left(\vec{a}_{D/A}\right)_t$, where the normal component evaluates to zero since $\omega_{AB} = 0$. Since we know all the directions, we can equate the expressions for $\vec{a}_D$ by components to find that $\alpha_{AB} = 1.28$rad/s^2.

To determine the acceleration of B, we have $\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_n + \left(\vec{a}_{B/A}\right)_t$, where the normal component is zero since $\omega_{AB} = 0$. But $\left(\vec{a}_{B/A}\right)_t = \overline{AB}\alpha_{AB}$. We can now substitute values to find that $\vec{a}_B = 0.6776$ft/s^2 at an angle of $61.82°$ South from East. ■

§20 November 17, 2017

§20.1

Example 20.1

The 18in radius flywheel is rigidly attached to a 1.5in radius shaft that can roll along parallel rails with a slope of 20°. Knowing that at the instant shown the centre of the shaft has a velocity of 1.2in/s and an acceleration of 0.5in/s^2, both directed down and to the left, determine the acceleration of the top of the wheel and the bottom of the wheel.

Solution. We first analyze the velocity, calling the centre of the wheel G, and the point of contact C, which is also the instantaneous center off rotation. Thus, $\vec{v}_C = 0$ and $\omega = \vec{v}_G/r = 1.2/1.5 = 0.8$rad/s. Now, we analyze acceleration. Since there is no slipping, we find that the velocity at the contact point is zero at all times. The acceleration however is not zero, since it is perpendicular to the rail, and $\vec{a}_C$ is directed 20° West from North. We can now write another expression for $\vec{a}_C$ using G as a reference so that $\vec{a}_C = \vec{a}_G + \left(\vec{a}_{C/G}\right)_n + \left(\vec{a}_{C/G}\right)_t$. We now equate both expressions and consider the components. Using the expressions for normal and tangential components of acceleration, we find that $\alpha = 0.333$rad/s counterclockwise.

We can now determine the acceleration at the top and the bottom. Expressing $\vec{a}_A$ using G as a reference, we can use the given value of $\vec{a}_G = 0.5$, and calculate the tangential and normal component values since we have α and ω. In particular, the tangential component is 18α and the normal component is $18\omega^2$. We can then find that $\vec{\alpha}_A = 13.36$in/s^2 at 61° South from West. Similarly, $\vec{\alpha}_B = 12.62$in/s^2 at 64° North from East. ■

§21 November 20, 2017

§22 Moments of Inertia

§23 November 22, 2017

§23.1 Parallel Axis Theorem

Theorem 23.1 (Parallel Axis Theorem)

Let AA' be an arbitrary axis, and BB' be a parallel centroidal axis d apart from AA'. The general relation between the moment of inertia I of the body with respect to AA' and its moment of inertia $\overline{I}$ with respect to BB' is

$$I = \overline{I} + md^2,$$

where m is the total mass of the body.

As a consequence, we find that for the same amount of material, placing the material far from the rotating axis increases the moment of inertia. Likewise, the moment of inertia about its centroidal axis is always the smallest.

§23.2 Angular Momentum of a Rigid Body

Consider a particle P_i of mass Δm_i from the original rigid body, where $\vec{r}_i$ is the position vector of this particle. Calculating the angular momentum of the particle about the origin O, we find that

$$\left(\vec{H}_o\right)_i = \vec{r}_i \times \Delta m_i \vec{v}_i.$$

Deriving this with respect to time, we obtain

$$\begin{aligned}\left(\vec{H}_o\right)_i' &= (\vec{r}_i)' \times \Delta m_i \vec{v}_i + \vec{r}_i \times \Delta m_i \left(\vec{v}_i\right)', \\ &= \vec{v}_i \times \Delta m_i \vec{v}_i + \vec{r}_i \times \Delta m_i \vec{a}_i, \\ &= 0 + \vec{r}_i \times \Delta m_i \vec{a}_i, \\ &= \vec{r}_i \times \sum \vec{F}_i, \\ &= \sum \left(\vec{M}_o\right)_i.\end{aligned}$$

For the entire rigid body, we now need to consider $\mathrm{d}m$ instead of Δm. Thus, the angular momentum of the rigid body is

$$\begin{aligned}\vec{H}_G &= \int (\vec{r})' \times (\vec{v})' \,\mathrm{d}m, \\ &= \int (\vec{r})' \times \left(\vec{\omega} \times (\vec{r})'\right) \mathrm{d}m, \\ &= \int \left(r'\right)^2 \vec{\omega} \mathrm{d}m, \\ &= \vec{\omega} \int \left(r'\right)^2 \mathrm{d}m, \\ &= I_G \vec{\omega}.\end{aligned}$$

Applying similar reasoning as in the case for a single particle, $\left(\vec{H}_G\right)' = \sum \vec{M}_G$. But then, we find that

$$\sum \vec{M}_G = I_G \vec{\alpha}.$$

Note that in order to apply the above equation, G must be the mass center.

§23.3 Equations of Motion for a Rigid Body

For a mass centre at G

§23.4 Plane Motion of a Rigid Body

The plane motion of a rigid body is completely determined by the resultant and moment resultant about the mass center G of the external forces.

$$\begin{aligned}\sum F_x &= m\bar{a}_x, \\ \sum F_y &= m\bar{a}_y, \\ \sum M_G &= \bar{I}\alpha.\end{aligned}$$

The external forces acting on a rigid body are equivalent to the effective forces of the various particles forming the body. This is known as *d'Alembert's Principle*. In general, the motion of a rigid body consists of a translation and a centroidal rotation.

Example 23.2

A 7.5kg rod BC of length 750mm connects a disk centered at A to crank CD. Crank CD of length 200mm is located at an angle of 30° East from North, with the rod attached to C at the top, and D is attached to the ground. Disk AB has the same slope, and is of radius 200mm. Knowing that the disk is made to rotate at the constant speed of 180rpm clockwise, determine for the position shown the vertical components of the forces exerted on rod BC by the pins at B and C.

Solution. We note that since it is rotating at a constant speed, we have $\alpha_A = 0$. The angular velocity of 180rpm can be expressed as 18.850rad/s. We first need to find the vertical components of the reactions at B and C. According to the kinematics, we have $(\vec{a}_B)_t = 0$ since the angular acceleration of the disk is zero.

$$\begin{aligned} a_B &= (a_B)_n \\ &= r\omega_A^2, \\ &= 0.2(18.850)^2, \\ &= 71.064\text{m/s}^2. \end{aligned}$$

We now draw the FBD on bar BC of the real forces on the left, and of the effective force on the right. Because we have two equivalent system, the sum of the moment on the left is the same as on the right. Taking the sum of moments at B, we find that $C_y = 193.99$N downwards. Summing forces in the vertical direction, we find that $B_y = 193.99$N downwards as well. ■

§24 November 24, 2017

§24.1

$$\begin{aligned} \sum F_x &= \left(\sum F_x\right)_{effective}, \\ \sum F_y &= \left(\sum F_y\right)_{effective}, \\ \sum M &= \left(\sum M\right)_{effective}. \end{aligned}$$

Example 24.1

A 20kg cabinet is mounted on casters that allow it to move freely ($\mu = 0$) on the floor. If a 100N force is applied at a distance of h from the floor horizontally to the right, determine the acceleration of the cabinet and the range of values for which the cabinet will not tip. The mass center is located 0.9m from the ground of the rectangular 0.6m wide cabinet.

Solution. We need to consider the possibility of rotation. We draw the free body diagram, with reactions at the casters A and B. The effective force is ma_G in the right direction. Note that since motion is impending only, we have $\alpha = 0$, implying that $I_G\alpha = 0$. Summing forces in the horizontal component with the effective horizontal forces, we find that $100 = 20a_G$. Thus, $\vec{a}_G = 5\text{m/s}^2$ to the right.

Now, consider tipping at point B located at the far end of the cabinet. This means that $R_A \approx 0$. Sum the moments at point B to find that $100h - 0.3mg = 0.9(ma_G)$. Substituting $m = 20$, $g = 9.81$, and $a_G = 5$, we find that $h = 1.489$. Thus, if h is larger than this value, the cabinet will rotate clockwise.

Now consider tipping at point A located at the near end of the cabinet. This means that $R_B \approx 0$. Sum the moments at point A to find that $100h + 0.3mg = 0.9(ma_G)$. Thus, we find that $h = 0.3114$. Thus, if h is smaller than this value, the cabinet will rotate counterclockwise. Thus, the conditions of h for which there is no tipping is $0.3114 \leq h \leq 1.489$. ■

Example 24.2

A beam AB of mass m and of uniform cross section is suspended from two springs. If the second spring on the right breaks, determine at that instant the angular acceleration of the beam, the acceleration of point A at the bottom of the first spring to the left, and the acceleration of point B on the right end of the beam. The first spring is attached to the top of point A at the left end of the beam. At $3L/4$ to the right, the second spring is attached to the top of the beam at C. Point B extends $L/4$ to the right of this. The beam is initially positioned horizontally.

Solution. Let G be the mass center of the beam. First, we need to determine the force in the springs before the second spring breaks. Since this is in equilibrium, we draw an FBD to determine the forces at A and C. We find that $A = mg/3$ and $C = 2mg3$.

At the moment that the second spring breaks, we are left with the real upwards force due to the spring at A and the downward force of gravity on the mass center. For the effective forces on the right hand side, we have an effective moment clockwise of I_G with an angular acceleration of α, and an effective force downwards at G of ma_G. At this instant, we have no velocity for beam AB and no new displacement Δy_A at A yet. Additionally, $A = mg/3$ at the instant the spring breaks.

By inspecting the left hand side, we can see immediately that the horizontal component of the effective force on the right hand side is zero. By summing the vertical forces on the left and the vertical effective forces on the right, we obtain

$$\frac{mg}{3} - mg = -ma_G,$$

so we can solve this to find that $a_G = g/3$ downwards (or $-g/3$). We can also sum the moments with respect to G to find that

$$\begin{aligned}\sum M_G &= \sum (M_G)_{effective}, \\ \frac{mg}{3}\left(\frac{L}{2}\right) &= I_G\alpha \\ &= \left(\frac{mL^2}{12}\right)\alpha.\end{aligned}$$

Solving this, we find that $\alpha = 2g/L$ in the clockwise direction.

From kinematics analysis, we find that the normal component of the acceleration of A with respect to G is zero since $\omega = 0$. Thus, $\left(\vec{a}_{A/G}\right)_t = \overline{AG}\alpha = \alpha L/2 = 2gL/2L = g$. Additionally, $\vec{a}_A = +\vec{a}_G + \vec{a}_{A/G} = -g/3 + g = g/3$ upwards. ■

§25 November 27, 2017

§25.1 Systems of Rigid Bodies

For two or more rigid bodies, free body diagrams can be drawn for the entire system, or for each rigid boy. External and internal forces relative to the free body diagram need to be considered. It is usually convenient to consider the mass centers of each rigid body instead of the mass center of the entire system.

§26 December 6, 2017

§26.1 Conservation of Energy (17.1E)

We use energy methods to determine the velocity. Note that the effective forces method relates to acceleration, whereas energy methods relate to velocity. The initial potential energy before the system is released from rest is $V_1 = (V_g)_1 = 0$, where we let this be the datum at which we define the potential energy to be zero. Since it is released from rest, the kinetic energy is $T_1 = 0$. The change in height Δh is the height change at the center of gravity. At the final position below the initial, we have a change of height, so $V_2 = (V_g)_2 = -mg\Delta h$. The kinetic energy at the final position is now $T_2 = m(v_G)^2/2 + I_G\omega^2/2$, where G denotes the mass center.

We may apply dependent motion relations to constrain v_G and ω. Finding the instantaneous center of rotation, we have $v_G = \overline{CG}\omega$.

Example 26.1

A 3kg slender rod rotates in a vertical plane about a pivot at B. The rod from A to C positioned horizontally is 750mm and from C to B where B is between A and C is 150mm. A spring of constant $k = 300\text{N/m}$ and of an undeformed length of 120mm is attached to the rod at end C. The other end of the spring is attached to D located 360mm below B. Knowing that in this position the rod has an angular velocity of 4rad/s clockwise, determine the angular velocity of the rod after it has rotated through 90° and 180°.

Solution. We place the datum for which potential energy is zero at the level of the bar. The moment of inertia is

$$\begin{aligned} I_G &= \frac{m\left(\overline{AC}\right)^2}{12} \\ &= \frac{3(0.75)^2}{12} \\ &= 0.140625 \end{aligned}$$

At the position shown, we have the mass center rotating around B, so $(v_G)_1 = \overline{GB}\omega = 0.225(4) = 0.9\text{m/s}$. Now, the deflection of the spring is $\sqrt{0.15^2 + 0.36^2} - 0.12 = 0.27\text{m}$. Thus, we find that the initial kinetic energy is

$$\begin{aligned} T_1 &= \frac{m(v_G)_1^2}{2} + \frac{I_G\omega^2}{2} \\ &= \frac{3(0.9)^2}{2} + \frac{0.140625(4)^2}{2} \\ &= 2.34\text{J} \end{aligned}$$

$$
\begin{aligned}
V_1 &= (V_g)_1 + (V_e)_1 \\
&= 0 + \frac{300(0.27)^2}{2} \\
&= 10.935\text{J}
\end{aligned}
$$

At the second position, the deflection is $0.36 - 0.15 - 0.12 = 0.09$m and $(v_G) = \overline{GB}\omega_2 = 0.225\omega_2$.

$$
\begin{aligned}
V_2 &= (V_g)_1 + (V_e)_1 \\
&= mg\Delta h + \frac{kx^2}{2} \\
&= 3(9.81)(0.225) + \frac{300(0.09)^2}{2} \\
&= 7.83675\text{J}
\end{aligned}
$$

$$
\begin{aligned}
T_2 &= \frac{m\,(v_G)_2^2}{2} + \frac{I_G\omega_2^2}{2} \\
&= \frac{3(0.225\omega_2)^2}{2} + \frac{0.140625(\omega_2)^2}{2} \\
&= 0.14625\omega_2^2\text{J}
\end{aligned}
$$

Now, we use the principle of conservation of energy to find $T_1 + V_1 = T_2 + V - 2$ to obtain $2.34 + 10.935 = 7.83675 + 0.14625\,(\omega_2)^2$. Solving this gives $\omega_2 = 6.0979$rad/s. Note that we do not know the sign of ω, since it could be rotating clockwise or counterclockwise.

Now that the rod has rotated 180°, we note that the gravitational potential energy and the elastic potential energy as the same as in the first question initially. That is, $(V_g)_1 = (V_g)_3$ and $(V_e)_1 = (V_e)_3$. ■